MATHEMATICS Proceedings A 89 (3), September 29, 1986

Classification results for contact forms

by Jesús Gonzalo

Departamento de Matematicas, Universidad de Murcia, Murcia, Spain

Communicated by Prof. W.T. van Est at the meeting of December 16, 1985

1. INTRODUCTION

A contact form is a 1-form ω on an odd-dimensional manifold M^{2n+1} such that $\omega \wedge (d\omega)^n$ is a volume form on M. Here $(d\omega)^n$ denotes the nth exterior power of $d\omega$.

If ω is contact, then so is $\varrho\omega$ for any nonvanishing function ϱ. Two contact forms ω and ω' are said to be equivalent when there is a diffeomorphism $f: M \to M$ satisfying $f^*\omega = \varrho\omega'$ for some ϱ.

For a sphere S^n, we will denote by $x_1, x_2, \ldots$ or $x, y, \ldots$ the restrictions to S^n of the coordinates in $\mathbb{R}^{n+1}$. For a circle S^1, we will use $d\theta$ to denote the canonical 1-form; then $d\theta = xdy - ydx$.

Here are three classical contact forms:

$$\omega_0 = x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3 \quad \text{on } S^3$$

$$\omega_0 = x d\theta + y dz - z dy \quad \text{on } S^1 \times S^2$$

$$\omega_1 = \cos\theta_3 d\theta_1 + \sin\theta_3 d\theta_2 \quad \text{on } T^3$$

they can be written as sums of two terms $f \cdot (gdh - hdg)$. This leads us to ask which compact 3-dimensional manifolds do have a contact form:

$$\omega = f_3 \cdot (f_1 df_2 - f_2 df_1) + f_6 \cdot (f_4 df_5 - f_5 df_4).$$

In the first part of this paper we answer that question: S^3, $S^1 \times S^2$, and T^3 are the only manifolds, but the standard forms, written above, are not the only possible ones. Indeed, for each of these manifolds we find an infinite sequence of such forms.

Since a form $\alpha = f\cdot(gdh - hdg)$ satisfies $\alpha\wedge d\alpha\equiv 0$, two is the minimal number of summands of this type that can make up a contact form in dimension 3. As a generalization, $n+1$ is the minimal number of summands of this type that can make up a contact form in dimension $2n+1$. We also have classical examples in this case, namely on the manifolds:

$$S^n\times T^{n+1}, S^{n+1}\times T^n, \dots, S^{2n+1}.$$

We prove in the second part that if M^{2n+1} is compact and if it has a contact form ω realizing the minimal number $n+1$ of summands $f\cdot(gdh - hdg)$, then M is diffeomorphic to one of the manifolds above, and, unlike the three-dimensional case, there is only one such contact form up to equivalence on each of those manifolds. We can say that the higher dimensional case is much simpler in this sense.

We stress the fact that one gets M up to diffeomorphism; in particular, an exotic sphere will not have such a contact form.

Also, the tori T^{2n+1} are not included in the list for $n\geq 2$. In fact, it has given a lot of work to find just an example of any contact form on T^5 (6), and its existence on T^7 is still an open question.

Numbers in parenthesis refer to bibliography.

2. THE THREE-DIMENSIONAL CASE

Our main purpose is to find the pairs (M,ω) where M is a 3-dimensional closed connected manifold and ω is a contact form on M, for which there exist smooth functions $f_1, f_2, \dots, f_6$: $M\to\mathbb{R}$ such that:

$$\omega = f_3\cdot(f_1 df_2 - f_2 df_1) + f_6\cdot(f_4 df_5 - f_5 df_4)$$

Let $\bar\omega_0$ denote the following 1-form on $\mathbb{R}^6$:

$$\bar\omega_0 = x_3\cdot(x_1 dx_2 - x_2 dx_1) + x_6\cdot(x_4 dx_5 - x_5 dx_4)$$

then we are looking for contact forms $\omega = f^*\bar\omega_0$, where $f: M\to\mathbb{R}^6$ is a smooth map.

Because of technical reasons, we will not classify the possible forms ω but the possible maps f, up to an equivalence relation that we define now.

NOTATION. Consider the differential operator:

$$T: C^\infty(M)\times C^\infty(M)\to\Lambda^1 M \quad (f,g)\mapsto fdg - gdf.$$

It is trivial to check that $T(\lambda f, \lambda g) = \lambda^2 T(f,g)$, for any $\lambda\in C^\infty(M)$.

Let λ, μ, ϱ be positive C^∞ functions on M, then from the contact form $\omega = f_3\cdot T(f_1, f_2) + f_6\cdot T(f_4, f_6)$ we pass to:

$$\varrho\omega = \frac{\varrho}{\lambda^2}\cdot f_3\cdot T(\lambda f_1, \lambda f_2) + \frac{\varrho}{\mu^2}\cdot f_6\cdot T(\mu f_4, \mu f_5).$$

Thus consider the following action of the multiplicative group $\mathbb{R}^+ \times \mathbb{R}^+ \times \times \mathbb{R}^+$ on $\mathbb{R}^6$:

$$\eta_{a,b,c}(x_1, x_2, x_3, x_4, x_5, x_6) = \left(ax_1, ax_2, \frac{c}{a^2} \cdot x_3, bx_4, bx_5, \frac{c}{a^2} \cdot x_6 \right).$$

DEFINITION. Two maps $f, g: M \to \mathbb{R}^6$ are equivalent if there exist a diffeomorphism $\varphi: M \to M$ and functions $a, b, c: M \to \mathbb{R}^+$ such that $g = \eta_{a,b,c}(f \circ \varphi)$.

The group η has the property $(\eta_{a,b,c})^* \bar{\omega}_0 = c\bar{\omega}_0$; it indeed arises from $\bar{\omega}_0$ in the following natural way. Let U be the open set:

$$U = \{x \in \mathbb{R}^6 | (\bar{\omega}_0 \wedge d\bar{\omega}_0)_x \neq 0\}$$

it consists of the sextuples x for which one at most of the following elements is zero: $(x_1, x_2), x_3, (x_4, x_5), x_6$.

Consider then $\bar{\omega}_0$ as a Pfaff equation on U, and its characteristic system [3]:

$$\left.\begin{array}{l} i_X \bar{\omega}_0 = 0 \\ i_X d\bar{\omega}_0 = \lambda \bar{\omega}_0 \end{array}\right\} \quad X \in TU$$

this defines a three-dimensional distribution D on U, generated by the following vector fields:

$$X_1 = x_1 \partial_{x_1} + x_2 \partial_{x_2} - 2x_3 \partial_{x_3}; \; X_2 = x_4 \partial_{x_4} + x_5 \partial_{x_5} - 2x_6 \partial_{x_6};$$

$$Y = x_3 \partial_{x_3} + x_6 \partial_{x_6}.$$

These vector fields are linearly independent at every point of U and they commute, giving a global 3-parameter group of diffeomorphisms of U:

$$\bar{\eta}_{s_1, s_2, t}(x) = (e^{s_1}x_1, e^{s_1}x_2, e^{t-2s_1}x_3, e^{s_2}x_4, e^{s_2}x_5, e^{t-2s_2}x_6).$$

Note that, calling: $e^{s_1} = a$ $e^{s_2} = b$ $e^t = c$, $\bar{\eta}_{s_1, s_2, t} = \eta_{a,b,c}$.

Given $x \in U$, the leaf of D passing through x is the η-orbit of x:

$$[x] = \{\eta_{a,b,c}(x) | a, b, c > 0\}$$

and the leaf space of D in U is the quotient U/η.

Since $\dim D = \dim \mathbb{R}^6 - \dim M$, it is trivial to see that given a map $f: M \to \mathbb{R}^6$ the form $f^* \bar{\omega}_0$ is contact if and only if $f(M) \subset U$ and f is an immersion of M into U transverse to D.

We are about to see that the leaf space U/η is a 3-manifold. Then $f: M \to U$ is an immersion transverse to D if and only if the composite map $M \xrightarrow{f} U \xrightarrow{[\;]} U/\eta$ is a local diffeomorphism.

Also, two maps $f, g: M \to U$ are equivalent in the definition given before if and only if there is a diffeo. $\varphi: M \to M$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[dr, "{[\;]\circ f}"] \arrow[d, "\varphi"'] & \\ M \arrow[r, "{[\;]\circ g}"'] & U/\eta \end{tikzcd}$$

so we classify local diffeomorphisms $M \to U/\eta$, where M is a closed, connected, 3-manifold.

The open set U is the union of two open sets: $U = U_1 \cap U_2$, each of them η-invariant:

$$U_1 = \{x \in U \mid x_3 \neq 0 \ x_6 \neq 0\}$$

$$U_2 = \{x \in U \mid (x_1, x_2) \neq (0,0) \ (x_4, x_5) \neq (0,0)\}.$$

CLAIM. The restrictions:

$$[\ ]: U_i \to U_i/\eta \quad i = 1, 2$$

are trivial fibrations. The fibre is $(\mathbb{R}^+)^3$.

Consider the submanifolds of $\mathbb{R}^6$:

$$V_1 = \{x \in \mathbb{R}^6 \mid x_1^2 + x_2^2 + x_4^2 + x_5^2 = 1 \ x_3^2 = x_6^2 = 1\} \approx S^0 \times S^0 \times S^3$$

$$V_2 = \{x \in \mathbb{R}^6 \mid x_1^2 + x_2^2 = x_4^2 + x_5^2 = x_3^2 + x_6^2 = 1\} \approx T^3$$

V_1 is contained in U_1, and V_2 is contained in U_2. For any $x \in U_1$, the condition $\eta_{a,b,c}(x) \in V_1$ gives three equations in a, b, c, whose solution is unique and C^∞ as functions of x in U_1:

$$c_1 = 1/((x_1^2 + x_2^2)|x_3| + (x_4^2 + x_5^2)|x_6|); \ a_1 = \sqrt{c|x_3|}; \ b_1 = \sqrt{c|x_6|}.$$

For any $x \in U_2$, the solution to $\eta_{a,b,c}(x) \in V_2$ is:

$$a_2 = 1/\sqrt{x_1^2 + x_2^2}; \ b_2 = 1/\sqrt{x_4^2 + x_5^2}; \ c_2 = 1/\sqrt{(x_1^2 + x_2^2)^2 x_3^2 + (x_4^2 + x_5^2)^2 x_6^2}$$

again C^∞ as functions on U_2. This proves the claim.

Let $\pi_i: U_i \to V_i$ be the map $\eta_{a_i(x), b_i(x), c_i(x)}(x)$. Consider also the intersections:

$$A = V_1 \cap U_2 = \{x \in \mathbb{R}^6 \mid x_1^2 + x_2^2 + x_4^2 + x_5^2 = 1; \ x_3^2 = x_6^2 = 1;$$
$$(x_1, x_2) \neq (0,0) \neq (x_4, x_5)\}$$

$$B = V_2 \cap U_1 = \{x \in \mathbb{R}^6 \mid x_1^2 + x_2^2 = x_4^2 + x_5^2 = x_3^2 + x_6^2 = 1; \ x_3 \neq 0; \ x_6 \neq 0\}$$

then the restrictions:

$$\varphi_1: \begin{matrix} B \to A \\ x \mapsto \pi_1(x) \end{matrix} \qquad \varphi_2: \begin{matrix} A \to B \\ x \mapsto \pi_2(x) \end{matrix}$$

are mutually inverse diffeomorphisms, and the pasting space $L = V_1 \bigcup_{\varphi_2} V_2$, or also $L = V_2 \bigcup_{\varphi_1} V_1$, together with the projection $\pi: U \to L$ induced by π_1 and π_2, is equivalent to the leaf space $[\ ]: U \to U/\eta$ of D in U.

So far we have proved:

PROPOSITION. *There is a pasting manifold $L = (S^0 \times S^0 \times S^3) \bigcup_{\varphi_2} T^3$ and a projection $\pi: U \to L$, such that a form $f^* \bar{\omega}_0$ is contact if and only if $f(M) \subset U$ and $\pi \circ f: M \to L$ is a local diffeomorphism.*

Description of the leaf manifold

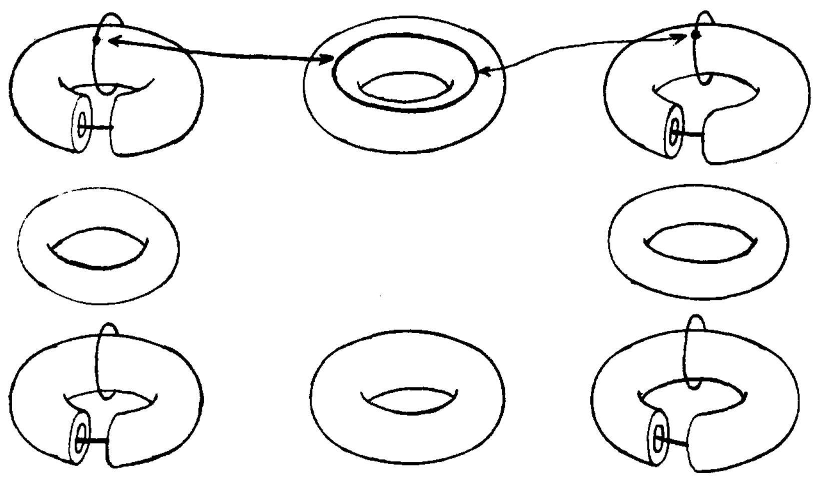
The manifold $V_2 \bigcup_{\varphi_1} V_1$ is compact, since V_1 and V_2 are compact. It is also connected, since V_2 is connected and $A = \varphi_1(B)$ intersects all the components of V_1. The joining open set A is $S^0 \times S^0 \times Q$, where Q is the subset of S^3: $\{(x_1, x_2, x_4, x_5) \in S^3 | x_1^2 + x_2^2 > 0 \quad x_4^2 + x_5^2 > 0\}$ diffeomorphic to $T^2 \times$(open interval).

The rest of the pasting manifold is:

$$V_1 - S^0 \times S^0 \times Q = S^0 \times S^0 \times ((0,0) \times S^1 \cup S^1 \times (0,0)) =$$
$$= \text{disjoint union of 8 copies of } S^1$$
$$V_2 - \varphi_2(S^0 \times S^0 \times Q) = T^3 - \{(x_3, x_6) \in S^1 | x_3 \neq 0 \; x_6 \neq 0\} \times T^2 =$$
$$= \{1, e^{\pi i/2}, e^{\pi i}, e^{3\pi i/2}\} \times T^2 =$$
$$= \text{disjoint union of 4 copies of } T^2.$$

This implies, in particular, that V_1 and V_2 are proper subsets of L, both open and compact, and L is thus not Hausdorff.

The following picture sketches the arrangement of the different parts in L:



Let $(\varepsilon_1, \varepsilon_2) \in S^0 \times S^0$ and consider a sequence in $\varepsilon_1 \times \varepsilon_2 \times Q$:

$$x^k = (x_1^k, x_2^k, \varepsilon_1, x_4^k, x_5^k, \varepsilon_2)$$

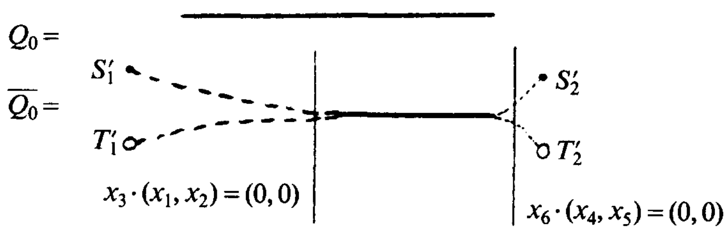
and suppose that it converges to $(0, 0, \varepsilon_1, x_4^0, x_5^0, \varepsilon_2)$ in $\varepsilon_1 \times \varepsilon_2 \times S^3$. We can choose the sequence $(x_1^k, x_2^k) \to (0,0)$ so that $\varphi_2(x^k)$ will converge to any point of the torus $(0, \varepsilon_2) \times T^2$ having the form $(0, \varepsilon_2) \times (x_1^0, x_2^0) \times (x_4^0, x_5^0)$. This means that the two points $(0, 0, \pm 1, x_4^0, x_5^0, \varepsilon)$ are Hausdorff inseparable from the points on the torus $(0;\varepsilon) \times T^2$: $(0, \varepsilon) \times (x_1, x_2) \times (x_4^0, x_5^0)$; these obviously form a circle. One example is indicated in the picture.

The same is true for $(x_1^0, x_2^0, \varepsilon, 0, 0, \pm 1)$ and the circle $(\varepsilon, 0) \times (x_1^0, x_2^0) \times (x_4, x_5)$ on the torus $(\varepsilon, 0) \times T^2$. Notice that the points $(x_1^0, x_2^0, \varepsilon, 0, 0, \pm 1)$ are Hausdorff separable from each other, and the same is true for $(0, 0, \pm 1, x_4^0, x_5^0, \varepsilon)$.

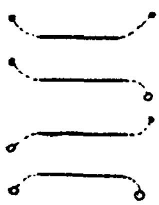
The closure of a component $Q_0=\varepsilon_1\times\varepsilon_2\times Q$ of $S^0\times S^0\times Q$ consists of Q_0 plus two circles and two tori:

$$S_1'=\varepsilon_1\times\varepsilon_2\times(0,0)\times S^1 \quad S_2'=\varepsilon_1\times\varepsilon_2\times S^1\times(0,0)$$
$$T_1'=(0,\varepsilon_2)\times T^2 \qquad\qquad T_2'=(\varepsilon_1,0)\times T^2.$$

It is thus a non-Hausdorff manifold with boundary. Schematically:



Here are the minimal compact subsets of L containing Q_0:



$S_1'\cup Q_0\cup S_2'=\varepsilon_1\times\varepsilon_2\times S^3$

$S_1'\cup Q_0\cup T_2'\approx S^1\times D^2$ here $D^2=\{(x,y)\in\mathbb{R}^2|x^2+y^2\le 1\}$

$T_1'\cup Q_0\cup S_2'\approx S^1\times D^2$

$T_1'\cup Q_0\cup T_2'\approx[0,1]\times T^2$

they are Hausdorff manifolds, three of them with boundary.

Finally, observe that S_1', S_2', T_1', T_2', are closed subsets of L.

The pieces of M

In what follows, C will denote a connected component of the set $(\pi\circ f)^{-1}(S^0\times S^0\times Q)$, that is the open subset of M given by $f_3\cdot(f_1,f_2)\neq(0,0)$ and $f_6\cdot(f_4,f_5)\neq(0,0)$. It is crucial to figure out the closure $\bar C$, for M is union of these closed subsets.

LEMMA 1. *The restriction $\pi\circ f: C\to\varepsilon_1\times\varepsilon_2\times Q$ is a finitely-sheeted covering map.*

PROOF. The points of $\varepsilon_1\times\varepsilon_2\times Q=Q_0$ satisfy the Hausdorff axiom with any point of L. Then for $y\in Q_0$ the set $(\pi\circ f)^{-1}(y)$ is closed. Since it is discrete, it is finite and so is $C\cap(\pi\circ f)^{-1}(y)$. One proves that $\pi\circ f: C\to Q_0$ is proper in the same way. This implies the lemma. QED

Let $\overline{Q_0}=S_1'\cup T_1'\cup Q_0\cup T_2'\cup S_2'$. Since S_i', T_i' are closed in L, so are the inverse images under $\pi\circ f$, thus compact submanifolds of M. Then the maps $\pi\circ f$: $S_1\to S_1'$ $S_1=a$ component of $(\pi\circ f)^{-1}(S_1')$, etc., are all compact coverings.

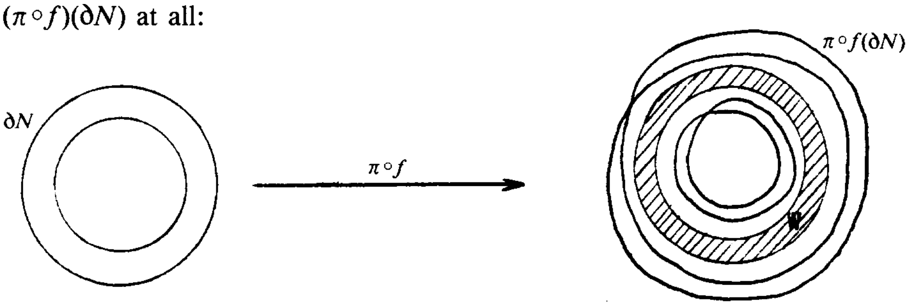
LEMMA 2. *The closure $\bar C$ consists of C plus a unique component of $f_3\cdot(f_1,f_2)=(0,0)$ and a unique component of $f_6\cdot(f_4,f_5)=(0,0)$. The map $\pi\circ f:\bar C\to\pi\circ f(\bar C)$ is a covering.*

PROOF. Because of the description made before of the minimal compact subsets of L containing $Q_0=\pi\circ f(C)$, the closure $\bar{C}$ must contain at least one component of $f_3\cdot(f_1,f_2)=(0,0)$ and one of $f_6\cdot(f_4,f_5)=(0,0)$.

Suppose, for instance, that $\bar{C}$ contains a component S_1 of $(f_1,f_2)=(0,0)$ and a component T_2 of $f_6=0$. We claim that $S_1\cup C\cup T_2$ is already compact, thus it is the whole closure of C. Similarly we can deal with the cases $\bar{C}\supset S_1\cup\cup C\cup S_2$, $\bar{C}\supset T_1\cup C\cup S_2$, and $\bar{C}\supset T_1\cup C\cup T_2$.

We have a homeomorphism: $h:S_1'\cup Q_0\cup T_2'\to D^2\times S^1$ taking S_1' to $(0,0)\times S^1$ and T_2' to $S^1\times\partial D^2$. This implies that if W is a tubular neighbourhood of S_1 (resp. T_2), arbitrarily small, and we take the reduced relative neighbourhood: $V=Q_0\cap W$, then the inclusion $V\hookrightarrow Q_0$ induces an isomorphism of the π_1 groups. Thus $((\pi\circ f)|C)^{-1}(V)$ is connected.

We now use the fact that S_1 and T_2 are compact. Let N be a compact, arbitrarily small, tubular neighbourhood of S_1 (resp. T_2). Then $(\pi\circ f)(\partial N)$ is an immersion of a compact manifold into L, not intersecting S_1' (resp. T_2'). Therefore there is a tubular neighbourhood W of S_1 (resp. T_2) not intersecting $(\pi\circ f)(\partial N)$ at all:



Then $((\pi\circ f)_{|C})^{-1}(W)$ is connected and it doesn't intersect ∂N, thus it is contained in N.

If (x_k) is a sequence in $S_1\cup C\cup T_2$, we can pass to a subsequence and get that $y_k=\pi\circ f(x_k)$ converges to some $y\in S_1'\cup Q_0\cup T_2'$. Three cases are possible:

a) $y\in Q_0$. Then, by lemma 1, (x_k) accumulates to some point $x\in C$.

b) $y\in S_1$. Take a sequence N_k of neighbourhoods of S_1, such that $\cap N_k=S_1$ and $y_k\in W_k$, for some neighbourhood W_k of S_1' satisfying $((\pi\circ f)_{|C})^{-1}(W_k)\subset\subset N_k$. Then $x_k\in N_k$ and x_k must accumulate to some $x\in S_1$.

c) $y\in T_2$. Exactly the same as b).

Now $S_1\cup C\cup T_2$ is a compact manifold with boundary and the map: $\pi\circ f:S_1\cup C\cup T_2\to S_1'\cup Q_0\cup T_2'$ is a local diffeomorphism that takes boundary to boundary and interior to interior. It is thus a covering. QED

REMARK. As a trivial consequence, given an open set $V=S\cup C$ or $V=C_1\cup\cup T\cup C_2$ the map $\pi\circ f:V\to\pi\circ f(V)$ is a covering.

The maps $M\to L$

THEOREM 1. *Let M^3 be a closed, connected manifold and $f:M\to U$ be a map inducing a contact form $\omega=f^*\bar{\omega}_0$. If $(f_1,f_2)\neq(0,0)\neq(f_4,f_5)$ on all of M, then the map $\pi\circ f$ is a compact, connected covering of the torus T^3.*

PROOF. We have $f(M) \subset U_2$, and $\pi \circ f : M \to V_2 = T^3$ is a local diffeomorphism. This is all we need. QED

REMARK. This implies that M is diffeomorphic to T^3.

We suppose now that either (f_1, f_2) or (f_4, f_5) has a zero $x_0 \in M$. This means that $\pi \circ f$ hits one of the eight circles that together with V_2 make up L.

By exchanging the summands $f_3 \cdot (f_1 df_2 - f_2 df_1)$ and $f_6 \cdot (f_4 df_5 - f_5 df_4)$ of ω, and then making changes like this: $f_3 \cdot (f_1 df_2 - f_2 df_1) = (-f_3) \cdot (f_2 df_1 - f_1 df_2)$, we can assume that $(f_4, f_5) = (0,0)$; $f_3 > 0$; $f_6 > 0$ at x_0, i.e. that $\pi \circ f(x_0)$ is on the circle $1 \times 1 \times S^1 \times (0,0)$.

THEOREM 2. *Let M^3, f be as in theorem 1, and let $x_0 \in M$ be a point such that $\pi \circ f(x_0)$ is on the circle $1 \times 1 \times S^1 \times (0,0)$. Then either of the following holds:*

I) *There exist an integer $m \geq 0$ and a diffeomorphism $\varphi : M \to S^3$ making the following diagram commutative:*

$$\begin{tikzcd} M \arrow[dr, "\pi\circ f"] \arrow[dd, "\varphi"'] & \\ & L \\ M \arrow[ur, "\pi\circ f_m"'] & \end{tikzcd}$$

where $f_m : S^3 \to \mathbb{R}^6$ is:

$$f_m(x_1, x_2, x_3, x_4) =$$

$$= \left(x_1, x_2, \sin\left(\frac{\pi}{4} + m\pi(x_1^2 + x_2^2)\right), x_3, x_4, \cos\left(\frac{\pi}{4} + m\pi(x_1^2 + x_2^2)\right)\right).$$

II) *There exist integers $m \geq 0$, $l \geq 1$, and a diffeomorphism, denoted $\varphi : M \to S^1 \times S^2$ such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[dr, "\pi\circ f"] \arrow[dd, "\varphi"'] & \\ & L \\ S^1 \times S^2 \arrow[ur, "\pi\circ f_{m,l}"'] & \end{tikzcd}$$

where $f_{m,l} : S^1 \times S^2 \to \mathbb{R}^6$ is:

$$f_{m,l}(\theta, x_1, x_2, x_3) = \left(x_1, x_2, \sin\left(\frac{\pi}{4} + \left(\frac{\pi}{2} + m\pi\right)\frac{x_3 + 1}{2}\right),\right.$$

$$\left.\cos(l\theta), \sin(l\theta), \cos\left(\frac{\pi}{4} + \left(\frac{\pi}{2} + m\pi\right)\frac{x_3 + 1}{2}\right)\right).$$

PROOF. For every integer k, let $A_k = \varepsilon_1 \times \varepsilon_2 \times Q$ where $(\varepsilon_1, \varepsilon_2) = \sqrt{2} \cdot e^{\pi i/4} \cdot i^k$. Also let $T(k) = e^{k\pi i/2} \times T^2$. Here k is not restricted to the values $0, 1, 2, 3$.

We will denote by C_k a component of $(\pi\circ f)^{-1}(A_k)$ in M, and by T_k a component of $(\pi\circ f)^{-1}(T(k))$ in M.

Let S_2 be the component of $(\pi\circ f)^{-1}(1\times 1\times S^1\times(0,0))$ passing through x_0. Looking at the neighbourhood of $1\times 1\times S^1\times(0,0)$ in L, we see that there is an open set $S_2\cup C_0$ in M, with C_0 unique, whose closure is either $S_2\cup C_0\cup S_1$ or $S_2\cup C_0\cup T_1$, by lemma 2.

In the first case, we have a covering: $S_2\cup C_0\cup S_1 \xrightarrow{\pi\circ f} 1\times 1\times S^3$; thus $S_2\cup C_0\cup S_1=M$ and $\pi\circ f$ is a diffeomorphism. The commutivity of the diagram:

$$\begin{tikzcd} M \arrow[rr, "\pi\circ f"] \arrow[d, "\varphi=\pi\circ f"'] & & L \\ S^3 \arrow[urr, "\pi\circ f_0"'] & & \end{tikzcd}$$

is trivial to check.

In the second case, we get a bigger set: $S_2\cup C_0\cup T_1\cup C_1$ by just looking at the neighbourhoods of $e^{\pi i/2}\times T^2=\pi\circ f(T_1)$ in L.

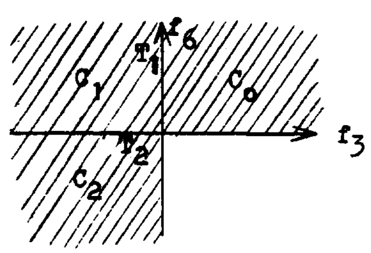
REMARK. The torus to be added to $S_2\cup C_0$ is a T_1, because it must be a component of $f_3\cdot(f_1,f_2)=(0,0)$ with $f_6>0$.

Taking closure on that open set, we add (say) a new torus:

$$S_2\cup C_0\cup T_1\cup C_1\cup T_2$$

and it can't be $\bar{C}_0\supset T_2$, since $\bar{C}_0=S_1\cup C_0\cup T_1$, so we add a new component: $S_2\cup C_0\cup T_1\cup C_1\cup T_2\cup C_2$.

Here is the distribution of the values of (f_3,f_6):



We keep adding components $T_k\cup C_k$, and the values of (f_3,f_6) keep wrapping counterclockwise around the origin in the plane. The manifold M gets completed when we finally add a circle.

For that last stage, we have two possibilities:

I) $M=S_2\cup C_0\cup T_1\cup C_1\cup\cdots\cup T_{2m}\cup C_{2m}\cup S_1 \quad m\geq 1$
where S_1 is a component of $(\pi\circ f)^{-1}(S^0\times S^0\times(0,0)\times S^1)$.

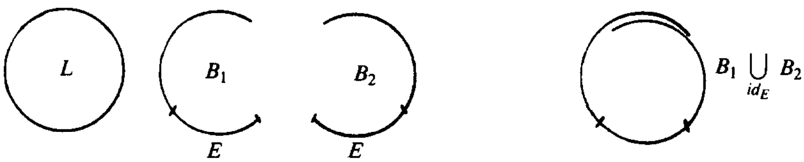
II) $M=S_2\cup C_0\cup T_1\cup C_1\cup\cdots\cup T_{2m+1}\cup C_{2m+1}\cup S_{2'} \quad m\geq 0$
where $S_{2'}$ is a component of $(\pi\circ f)^{-1}(S^0\times S^0\times S^1\times(0,0))$.

We make use now of the following:

PROPERTY. If:

$u: M \to L$ is a local diffeomorphism
G_1, G_2 are open subsets of M
B_1, B_2 are open subsets of L
$u: G_i \to B_i$ is a covering $i=1,2$
$G_1 \cap G_2$ is saturated: $(u|G_1 \cup G_2)^{-1}(u(G_1 \cap G_2)) = G_1 \cap G_2$
and we call $E = u(G_1 \cap G_2)$

then $G_1 \cup G_2$ is a covering space of $B_1 \bigcup\limits_{id_E} B_2$:



the map being $\tilde{u} = (u_{|G_1}) \bigcup\limits_{v} (u_{|G_2})$, where: v = identity of the graph of $u|G_1 \cap G_2$.

Furthermore, there is an obvious projection $P: B_1 \bigcup\limits_{id_E} B_2 \to B_1 \cup B_2$ such that $P \circ \tilde{u} = u|G_1 \cup G_2$.

CASE I. The manifold M is a covering space of:

$$L_{2m} = (1 \times 1 \times S^1 \times (0,0) \cup A_0) \bigcup_{id_{A_0}} (A_0 \cup T(1) \cup A_1) \bigcup_{id_{A_1}} \cdots \bigcup_{id_{A_{2m}}} (A_{2m} \cup \varepsilon_1 \times \varepsilon_2 \times (0,0) \times S^1)$$

by a map π_f, and there is a canonical projection $P_{2m}: L_{2m} \to L$ such that $P_{2m} \circ \pi_f = \pi \circ f$. It is trivial to see that $L_{2m} \approx S^3$, thus π_f is a diffeomorphism.

Let N, g be another pair realizing de same decomposition as M, f:

$$N = S_2' \cup C_0' \cup T_1' \cup C_1' \cup \cdots \cup T_{2m}' \cup C_{2m}' \cup S_1'.$$

Since π_f and π_g are diffeomorphisms, there exists a diffeomorphism denoted $\varphi: M \to N$ making commutative the diagram:

$$\begin{tikzcd} M \arrow[dr, "\pi_f"] \arrow[d, "\varphi"'] & & \\ N \arrow[r, "\pi_g"'] & L_{2m} \arrow[r, "P_{2m}"] & L \end{tikzcd}$$

then:

$$\begin{tikzcd} M \arrow[dr, "\pi \circ f"] \arrow[d, "\varphi"'] & \\ N \arrow[r, "\pi \circ g"'] & L \end{tikzcd}$$

is trivially commutative.

As a consequence of this digression, we only have to check that for the maps $f_m: S^3 \to U$ given in the statement of the theorem the form $f_m^* \bar{\omega}_0$ is contact and the resulting decomposition of S^3 is the one given here, but that is straightforward.

CASE II. The manifold M is a covering space of:

$$L_{2m+1} = (1 \times 1 \times S^1 \times (0,0) \cup A_0) \bigcup_{id_{A_0}} (A_0 \cup T(1) \cup A_1) \bigcup_{id_{A_1}} \cdots \bigcup_{id_{A_{2m+1}}}$$

$$(A_{2m+1} \cup \varepsilon_1 \times \varepsilon_2 \times S^1 \times (0,0))$$

by a map π_f, and we have a canonical projection $P_{2m+1}: L_{2m+1} \to L$ such that $P_{2m+1} \circ \pi_f = \pi \circ f$. It is trivial to see that $L_{2m+1} \approx S^1 \times S^2$.

Let N, g, be another pair realizing the same decomposition:

$$N = S_2' \cup C_0' \cup T_1' \cup C_1' \cup \cdots \cup T_{2m+1}' \cup C_{2m+1}' \cup S_{2'}'$$

and also: degree (π_f) = degree (π_g). By the classification of the coverings of $S^1 \times S^2$, there exists a diffeomorphism $\varphi: M \to N$ such that we have $\pi_g \circ \varphi = \pi_f$, and we conclude $(\pi \circ f) \circ \varphi = \pi \circ g$ as before.

It is again straightforward to check that for $f_{m,l}: S^1 \times S^2 \to U$ the form $f_{m,l}^* \bar{\omega}_0$ is contact, that the decomposition of $S^1 \times S^2$ that arises is the one given here and that $\pi_{f_{m,l}}$ has degree l. QED

We can now state the following

THEOREM A. *Consider pairs (M, ω) where M is a closed, connected, 3-manifold; and $\omega = f^* \bar{\omega}_0$ is a contact form on M for some map $f: M \to \mathbb{R}^6$. For any such pair, there is a term in one of the sequences below:*

$$(S^3, \omega_m)\ m \geq 0; \ (S^1 \times S^2, \omega_m)\ m \geq 0; \ (T^3, \omega_m)\ m \geq 1$$

equivalent to (M, ω) in the sense of the definition given in the introduction. The sequences are defined as follows:

$$\omega_m = \sin\left(\frac{\pi}{4} + m\pi(x_1^2 + x_2^2)\right)(x_1 dx_2 - x_2 dx_1) +$$

$$+ \cos\left(\frac{\pi}{4} + m\pi(x_1^2 + x_2^2)\right)(x_3 dx_4 - x_4 dx_3) \text{ for } S^3$$

$$\omega_m = \sin\left(\frac{\pi}{4} + \left(\frac{\pi}{2} + m\pi\right)(x_3 + 1)/2\right)(x_1 dx_2 - x_2 dx_1) +$$

$$+ \cos\left(\frac{\pi}{4} + \left(\frac{\pi}{2} + m\pi\right)(x_3 + 1)/2\right) d\theta \text{ for } S^1 \times S^2$$

$$\omega_m = \cos(m\theta_3) d\theta_1 + \sin(m\theta_3) d\theta_2 \text{ for } T^3.$$

PROOF. That the possible manifolds are S^3, $S^1 \times S^2$ or T^3, is an obvious consequence of theorems 1 and 2.

For $M = S^3$, the equality $\pi \circ f = \pi \circ f_m \circ \varphi$ implies that $f^*\bar{\omega}_0$ is a scalar multiple of the form $\varphi^* f_m^* \bar{\omega}_0$, but $f_m^* \bar{\omega}_0 = \omega_m$ for S^3.

For $M = S^1 \times S^2$, we reduce similarly to the forms $f_{m,l}^* \bar{\omega}_0$. But we have the identity: $\cos(l\theta) d \sin(l\theta) - \sin(l\theta) d \cos(l\theta) = d(l\theta) = l d\theta$, thus

$$f_{m,l}^* \bar{\omega}_0 = \sin\left(\frac{\pi}{4} + \left(\frac{\pi}{2} + m\pi\right)(x_3 + 1)/2\right)(x_1 dx_2 - x_2 dx_1) + l \cdot \cos(\cdots) d\theta,$$

and it is trivial to see that this is equivalent to $f_{m,1}^* \bar{\omega}_0 = \omega_m$.

For $M = T^3$, we have the covering map $\pi \circ f : M \to T^3$ and if $j : T^3 \hookrightarrow U$ is the inclusion (recall $T^3 = V_2$) we have $j^* \bar{\omega}_0 = \omega_1$. Then $\omega = f^* \bar{\omega}_0$ is a scalar multiple of $(j \circ \pi \circ f)^* \bar{\omega}_0 = (\pi \circ f)^* \omega_1$.

In this way, we restrict ourselves to the case $\omega = h^* \omega_1$ where the map $h : T^3 \to T^3$ is a covering. We have $h = R \circ \Phi$ where $\Phi : T^3 \to T^3$ is a diffeomorphism and R is a standard covering of T^3. That means there is a lower triangular matrix of integers:

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & m \end{pmatrix}$$

whose diagonal entries are positive and such that:

$$R(z_1, z_2, z_3) = (z_1^{a_{11}} \cdot z_2^{a_{21}} \cdot z_3^{a_{31}}, z_2^{a_{22}} \cdot z_3^{a_{32}}, z_3^m).$$

Since $h^* \omega_1 = \Phi^* R^* \omega_1$ is isomorphic to $R^* \omega_1$, we only have to show that $R^* \omega_1$ is equivalent to ω_m.

It is trivial to check that:

$$R^* \omega_1 = a_{11} \cos(m\theta_3) d\theta_1 + (a_{21} \cos(m\theta_3) + a_{22} \sin(m\theta_3)) d\theta_2 + dw(\theta_3)$$

where

$$w(\theta) = \frac{1}{m} a_{31} \sin(m\theta) - \frac{1}{m} a_{32} \cos(m\theta).$$

Consider now the forms:

$$\omega(t) = a_{11} \cos(m\theta_3) d\theta_1 + (a_{21} \cos(m\theta_3) + a_{22} \sin(m\theta_3)) d\theta_2 + t dw(\theta_3)$$

it is straightforward to check that $\omega(t) \wedge d\omega(t) = \omega(0) \wedge d\omega(0)$ is constant, then all $\omega(t)$ are contact because $\omega(1)$ is. By Gray's stability theorem [5], $R^* \omega_1$ is isomorphic to a scalar multiple of $\omega(0)$. It is only left to show that $\omega(0)$ is equivalent to ω_m.

The easiest way to do that is to take the map that induces $\omega(0)$:

$$f'(\theta_1, \theta_2, \theta_3) = (\cos\theta_1, \sin\theta_1, a_{11} \cos(m\theta_3),$$
$$\cos\theta_2, \sin\theta_2, a_{21} \cos(m\theta_3) + a_{22} \sin(m\theta_3))$$

and to go back to the proof of theorem 1. We get a covering $\pi \circ f': T^3 \to T^3$ equivalent to the covering $h_m(z_1, z_2, z_3) = (z_1, z_2, z_3^m)$, thus $\omega(0)$ is equivalent to $h_m^* \bar{\omega}_0 = \omega_m$. QED

COROLLARY 1. *Let M be a closed, connected 3-manifold, with a contact form $\omega = f_3 \cdot (f_1 df_2 - f_2 df_1) + f_6 \cdot (f_4 df_5 - f_5 df_4)$, where the f_i's are C^∞ functions on M. Then there are three possible cases:*

1) *If $f_1^2 + f_2^2 > 0$ and $f_4^2 + f_5^2 > 0$ everywhere, then $M \approx T^3$ and there is an integer $m \geq 1$ such that the sets $f_3 = 0$ and $f_6 = 0$ are each union of $2m$ 2-dimensional tori (thus $f_3 \cdot f_6 = 0$ consists of $4m$ tori) and ω is equivalent to ω_m.*

2) *If $f_4^2 + f_5^2 > 0$ everywhere, and $f_1^2 + f_2^2$ vanishes along two circles (up to exchanging (f_1, f_2, f_3) with (f_4, f_5, f_6)), then $M \approx S^1 \times S^2$ and there is an integer $m \geq 0$ such that $f_6 = 0$ is union of $m+1$ tori and $f_3 = 0$ is union of m tori. Then ω is equivalent to ω_m.*

3) *Both $f_1^2 + f_2^2$ and $f_4^2 + f_5^2$ vanish, each along a different circle. Then the manifold M is diffeomorphic to S^3 and there is an integer $m \geq 0$ such that $f_3 = 0$ and $f_6 = 0$ both consist of m tori (totalling $2m$ tori), and the form ω is equivalent to ω_m.*

REMARK. The number of elements in $\{f_1^2 + f_2^2, f_4^2 + f_5^2\}$ that have a zero determines the manifold, whereas the number of vanishing tori for f_3 and f_6 determines the equivalence class of the form.

PROOF OF COROLLARY. The only non-obvious thing is the number of vanishing tori in case 3).

Now, $\pi \circ f$ is essentially the covering $R: T^3 \to T^3$ given by the triangular matrix, and it is trivial to see that for R both f_3 and f_6 vanish along $2m$ tori. QED

There remains the question of the existence of equivalences between terms in a given sequence: ω_m and ω_{m+p}. Something is known in the case of the 3-sphere.

LUTZ'S METHOD [5]. In the proof of theorem 2 we said it is straightforward to check that all forms ω_m are contact. Indeed, one sees they all define the same orientation. Now suppose $\varphi: S^3 \to S^3$ is a diffeomorphism satisfying $\varphi^* \omega_{m+p} = \varrho \omega_m$ for some nonvanishing function ϱ. Then $\varphi^*(\omega_{m+p} \wedge d\omega_{m+p}) = \varrho^2 \omega_m \wedge d\omega_m$, but since ϱ^2 is positive $\omega_{m+p} \wedge d\omega_{m+p}$ defines the same orientation as $\varrho^2 \omega_m \wedge d\omega_m$. By Cerf's theorem (2), there exists an isotopy φ_t taking φ to the identity.

That means that ω_m is isotopic to ω_{m+p} through contact forms. Then fix a global coframe $\{\theta^1, \theta^2, \theta^3\}$ for T^*S^3 and for any nonvanishing 1-form ω on S^3 let $f\omega: S^3 \to S^2$ be the map associated to the coefficient functions of ω in that coframe. We get that ω is homotopic to ω' through nonvanishing 1-forms if and only if f_ω is homotopic to $f_{\omega'}$.

On the other hand, a map $S^3 \to S^2$ and its antipode are always homotopic, thus ω is always homotopic to $-\omega$; in other words: the homotopy class of f_ω determines the homotopy class of the line bundle spanned by ω and conversely.

In summary, for ω_m to be equivalent to ω_{m+p} it is necessary that the associated maps f_{ω_m} and $f_{\omega_{m+p}}$ be homotopic. It is just a computation to see that $f_{\omega_m} \sim f_{\omega_{m+p}}$ if and only if $m \equiv m+p (\text{mod. } 2)$, so we get two sequences: $(\omega_{2k})_{k \geq 0}$ $(\omega_{2k+1})_{k \geq 0}$ no term in the first being equivalent to any term in the second.

BENNEQUIN'S METHOD [1]. Bennequin has proved in his thesis that if γ is any unknotted simple closed integral curve of ω_0 in S^3 and if γ' denotes the result of translating γ slightly along its normals in the planes Ker ω_0, then γ and γ' are linked. The linking number is the same for the two possible choices of the normal vector field along γ.

Now, for $m \geq 1$, the curve in

$$S^3 : \gamma_m(t) = \left(\sqrt{\frac{\pi}{4m}} \cos t, \sqrt{\frac{\pi}{4m}} \sin t, \sqrt{1 - \frac{\pi}{4m}}, 0 \right)$$

is an unknotted simple closed integral curve of ω_m, bounding the disc:

$$D^2 = \left\{ (x_1, x_2, x_3, x_4) | x_1^2 + x_2^2 \leq \frac{\pi}{4m} \; x_4 = 0 \right\}$$

and one of the normal translates of γ_m does not intersect D^2 at all. Therefore ω_0 is not equivalent to ω_{2m} for $m \geq 1$.

3. THE HIGHER DIMENSIONAL CASE

Let M be a $(2n+1)$-dimensional manifold and let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be 1-forms on M satisfying $\alpha_i \wedge d\alpha_i \equiv 0$. It is trivial to see that a form $\omega = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ cannot be contact unless $k \geq n+1$. On the other hand, a form $\alpha = f_3 \cdot (f_1 df_2 - f_2 df_1)$ is integrable since it is proportional to $f_1 df_2 - f_2 df_1$, which is integrable because it is expressed with only two functions. Thus a form

$$\omega = \sum_{i=1}^{k} f_{3i} \cdot (f_{3i-2} df_{3i-1} - f_{3i-1} df_{3i-2})$$

can be contact in dimension $2n+1$ only if $k \geq n+1$.

We want to find the pairs (M^{2n+1}, ω) that realize the minimum $k = n+1$. The case $n = 1$ we just studied, now we are interested in $n \geq 2$.

The following manifolds:

$$S^{2n+1}, S^1 \times S^{2n}, S^1 \times S^1 \times S^{2n-1}, \ldots, (S^1)^{n+1} \times S^n$$

realize that situation with their standard contact forms. Namely, the classical contact form on $T^p \times S^{n+q}$, here $p+q = n+1$, is:

$$\omega = \sum_{i=1}^{p} y_i d\theta_i + \sum_{j=1}^{q} (y_{p+2j-1} dy_{p+2j} - y_{p+2j} dy_{p+2j-1}).$$

Like for $n=1$, we let $\bar{\omega}_0$ denote the following 1-form on $\mathbb{R}^{3(n+1)}$:

$$\bar{\omega}_0=\sum_{i=1}^{n+1} x_{3i}(x_{3i-2}dx_{3i-1}-x_{3i-1}dx_{3i-2})$$

and we are looking for contact forms $\omega=f^*\bar{\omega}_0$ where $f:M\to\mathbb{R}^{3(n+1)}$ is a smooth map. The result we will reach is then the following:

THEOREM B. *Consider pairs (M^{2n+1},ω) where M is closed, connected, and where $\omega=f^*\bar{\omega}_0$ is a contact form on M. For $n\geq 2$, the only possible manifolds are $T^p\times S^{n+q}$, $(p+q=n+1)$, and the only possible forms, up to scalar factor and isomorphism, are the standard ones.*

Thus we don't get a sequence of possible forms as in dimension 3.

We are going to use the same setup as for $\dim=3$.

The open set: $U=\{x\in\mathbb{R}^{3(n+1)}|(\bar{\omega}_0\wedge(d\bar{\omega}_0)^n)_x\neq 0\}$ consists of the points $x=(x_1,\ldots,x_{3n+3})$ such that each of the sets:

$$\{(x_1,x_2),x_3\},\ldots,\{(x_{3n+1},x_{3n+2}),x_{3n+3}\}$$

has a nonzero element, and at least one of them has both elements nonzero. The linear system:

$$\left.\begin{aligned} i_X\bar{\omega}_0&=0\\ i_Xd\bar{\omega}_0&=\lambda\bar{\omega}_0\end{aligned}\right\}\quad X\in TU$$

defines an $(n+2)$-dimensional distribution D on U, generated by the following vector fields:

$$X_i=x_{3i-2}\partial_{x3i-2}+x_{3i-1}\partial_{x3i-1}-2x_{3i}\partial_{x3i}\quad 1\leq i\leq n+1$$

$$Y=\sum_{i=1}^{n+1} x_{3i}\partial_{x3i}$$

which are linearly independent at every point and commute with one another, thus generating a global action of the additive group $\mathbb{R}^{n+2}$ on U:

$$\eta_{s_1,\ldots,s_{n+1},t}(x)=(e^{s_1}x_1,e^{s_2}x_2,e^{t-2s_1}x_3,\ldots).$$

If we substitute $s_1,\ldots,s_{n+1},t$ by $a_1=e^{s_1},a_2=e^{s_2},\ldots,$ we get an action of the multiplicative group $(\mathbb{R}^+)^{n+2}$ on U, also denoted by η:

$$\eta_{a_1,\ldots,a_{n+1},b}(x)=\left(a_1x_1,a_1x_2,\frac{b}{a_1^2}x_3,\ldots\right).$$

The leaf of D in U passing through a point x equals the η-orbit of x:

$$[x]=\{\eta_{a_1,\ldots,a_{n+1},b}(x)|a_1,\ldots,a_{n+1},b\in\mathbb{R}^+\}$$

so the leaf space of D is U/η. We want to describe it as a pasting space.

For every subset $I \subset \{1, \ldots, n+1\}$, let V_I be the following submanifold of $\mathbb{R}^{3(n+1)}$:

$$\left.\begin{array}{ll} x_{3i-2}^2 + x_{3i-1}^2 = 1, & i \in I \\ x_{3j}^2 = 1 & , \ j \notin I \\ \sum_{i \in I} x_{3i}^2 + \sum_{j \notin I} (x_{3j-2}^2 + x_{3j-1}^2) = 1 & \end{array}\right\}$$

it is contained in U, and equals $(S^0)^q \times T^p \times S^{n+q}$, where $p = \neq(I)$ $q = n+1-p$.

Define

$$U_I = \{x \in U \,|\, [x] \cap V_I \neq \emptyset\} =$$
$$= \{x \in U \,|\, (x_{3i-2}, x_{3i-1}) \neq (0,0) \ i \in I; \quad x_{3j} \neq 0 \quad j \notin I\}.$$

Then U_I is open and for $x \in U_I$ the intersection $[x] \cap V_I$ consists of a unique point $\eta_{a_I(x), b_I(x)}(x)$, where $a_I : U_I \to (\mathbb{R}^+)^{n+1}$ $b_I : U_I \to \mathbb{R}^+$ are smooth functions on U_I. Since U_I is obviously η-invariant, this implies that the maps $U_I \to U_I/\eta$ are trivial fibrations with base V_I and fibre $(\mathbb{R}^+)^{n+2}$. Since $\bigcup_I U_I = U$, the leaf space $U \to U/\eta$ is equivalent to the pasting space L of the V_I's using the open sets $V_{IJ} = V_I \cap U_J$ and the diffeomorphisms

$$\varphi_{IJ} : \begin{array}{l} V_{IJ} \to V_{JI} \\ x \mapsto [x] \cap V_J \end{array}$$

which obviously satisfy the condition $\varphi_{KJ} \circ \varphi_{JI} = \varphi_{KI}$ on $V_I \cap U_J \cap U_K$.

Given the natural projection $\pi : U \to L$, we know that $f^*\bar{\omega}_0$ is contact if and only if $f(M) \subset U$ and $\pi \circ f : M \to L$ is a local diffeomorphism.

The manifold $V_{\{1,\ldots,n+1\}}$ is $T^{n+1} \times S^n$, and the open set $\pi(\bigcap_I U_I)$ equals: $T^{n+1} \times \{(y_1, \ldots, y_{n+1}) \in S^{n+1} | y_1 \neq 0, \ldots, y_{n+1} \neq 0\}$, and it can be identified to $(S^0)^{n+1} \times T^{n+1} \times I_n$, where I_n is the open n-cell:

$$\{(y_1, \ldots, y_{n+1}) \in S^n | y_1 > 0, \ldots, y_{n+1} > 0\}.$$

Like for the case $n=1$, we call $Q = T^{n+1} \times I_n$, and will denote by C any connected component of $(\pi \circ f)^{-1}((S^0)^{n+1} \times Q)$ in M, i.e. any maximal open connected set where all the elements:

$$f_3 \cdot (f_1, f_2), \ldots, f_{3n+3} \cdot (f_{3n+1}, f_{3n+2})$$

are nonzero.

Also like for $n=1$, the pasting space L equals $(S^0)^{n+1} \times Q$ plus lower dimensional submanifolds, and there are many pairs of points Hausdorff inseparable. The points of a pair lying on two different lower dimensional submanifolds. Then the points of $(S^0)^{n+1} \times Q$ satisfy the Hausdorff axiom with any point of L, and we get:

LEMMA 1′. *The restrictions $\pi \circ f : C \to \varepsilon_1 \times \cdots \times \varepsilon_{n+1} \times Q$ are finitely-sheeted coverings.*

The proof is the same as for lemma 1.

It is again crucial to figure out the structure of the closure $\bar{C}$.

LEMMA 2′. *The closure $\bar{C}$ consists of C plus a unique component of the submanifold:*

$$\left.\begin{array}{ll} f_{3i}\cdot(f_{3i-2},f_{3i-1})=(0,0) & i\in I \\ f_{3j}\cdot(f_{3j-2},f_{3j-1})\neq(0,0) & j\notin I \end{array}\right\}$$

for every proper subset $I\subsetneq\{1,\dots,n+1\}$. Call this unique component S_I. There exists a set $I_0\subset\{1,\dots,n+1\}$ such that for every $I\subsetneq\{i,\dots,n+1\}$ we have:

$$\left.\begin{array}{ll} f_{3i}=0 & i\in I\cap I_0 \\ (f_{3i-2},f_{3i-1})=(0,0) & i\in I-I_0 \end{array}\right\} \text{ on } S_I \qquad (*)$$

Finally, the restriction $\pi\circ f:\bar{C}\to\pi\circ f(\bar{C})$ is a covering.

PROOF. It is not difficult to see that the minimal compact subsets of L containing $Q_0=\varepsilon_1\times\cdots\varepsilon_{n+1}\times Q$ are those of the form:

$$Q_0\cup\bigcup_{I\subsetneq\{1,\dots,n+1\}} S_I'$$

where S_I' is a component of the image under π of those $x\in U$ satisfying:

$$\left.\begin{array}{ll} x_{3i}=0 & i\in I\cap I_0 \\ (x_{3i-2},x_{3i-1})=(0,0) & i\in I-I_0 \\ x_{3j}\cdot(x_{3j-2},x_{3j-1})\neq(0,0) & j\notin I \end{array}\right\} \text{ plus } \varepsilon_i x_{3i}\geq 0 \quad \forall i.$$

so we have a compact minimal subset for every $I_0\subset\{1,\dots,n+1\}$.

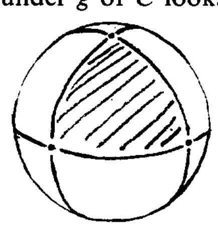
The compatibility conditions (*), among the submanifolds in $\bar{C}$, do not exist in dimension three because one of the sets: $I\cap I_0$, $I-I_0$, is always empty.

But the important difference with the three-dimensional case is that in such case the pieces S_I are all compact, whereas now many of them are not necessarily compact. Before we continue, let us illustrate with an example what is going on.

Let $g:M\to S^n$ be the following map:

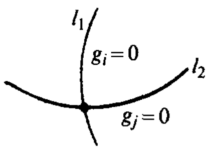
$$\frac{(f_3\cdot(f_1^2+f_2^2),\dots,f_{3n+3}\cdot(f_{3n+1}^2+f_{3n+2}^2))}{\sqrt{\sum_{i=1}^{n+1} f_{3i}^2\cdot(f_{3i-2}^2+f_{3i-1}^2)^2}}.$$

Let $n=2$, the image under g of $\bar{C}$ looks like this:

$C\subset M \xrightarrow{g}$ 

The arcs correspond to components $g_i=0$. The dots correspond to components of $g_i=g_j=0$.

The compatibility conditions (*) arise from the fact that $g_i=0$ and $g_j=0$ actually intersect:



Thus if $f_{3i}=0$ for the arc l_1 and $(f_{3j-2}, f_{3j-1})=(0,0)$ for the arc l_2, it must be: $f_{3i}=0$ $(f_{3j-2}, f_{3j-1})=(0,0)$ for $l_1 \cap l_2$.

It is important to observe that in dimension 3 the boundary ∂C always has two components while for $n \geq 2$ it is always going to be connected.

This is crucial for the uniqueness of the form ω for each of the manifolds $T^p \times S^{n+q}$, because an "unfolding" of L, like the spaces L_{2m} and L_{2m+1} of the three-dimensional case, cannot exist.

We can guess from the picture that components like $g_1=0$ $g_2 \neq 0$ $g_3 \neq 0$ are not compact, only components like $g_1=0$ $g_2=0$ $g_3 \neq 0$ are. We also see that $\bar{C}$ is a manifold with boundary and corners. For the $(2n+1)$-dimensional case, $\bar{C}$ is a manifold locally modelled on $[0,\infty)^k \times \mathbb{R}^{2n+1-k}$, for $0 \leq k \leq n$. Finally, the picture shows that a manifold like $g_1=0$ (no restrictions on g_2, g_3) is compact and intersects $\bar{C}$, but unlike the case $n=1$ it is not totally contained in $\bar{C}$.

For every $i \in \{1,\dots,n+1\}$, choose a component S_i of $g_i=0$ $g_j \neq 0$ $\forall j \neq i$ that intersects $\bar{C}$, it exists for all i. Can assume the indices $1,\dots,n+1$ reassigned so that:

for $1 \leq i \leq p$, S_i is a component of $f_{3i}=0$ $g_j \neq 0$ $\forall j \neq i$

for $p+1 \leq i \leq n+1$, S_i is a component of $(f_{3i-2}, f_{3i-1})=(0,0)$

$g_j \neq 0$ $\forall j \neq i$.

Also, let:

$$Q_0=\pi \circ f(C)=\varepsilon_1 \times \cdots \times \varepsilon_{n+1} \times Q$$

$$S_i'=\pi(\{x \in U \mid x_{3i}=0;\ \varepsilon_j x_{3j}>0 \text{ and } (x_{3j-2}, x_{3j-1}) \neq (0,0)\ \forall j \neq i\})$$
$$1 \leq i \leq p$$

$$S_i'=\pi(\{x \in U \mid (x_{3i-2}, x_{3i-1})=(0,0);\ \varepsilon_j x_{3j}>0\ \forall j;\ (x_{3j-2}, x_{3j-1}) \neq (0,0)$$
$$\forall j \neq i\})\ p+1 \leq i \leq n+1$$

$$V_i'=\pi(\{x \in U \mid x_{3i}=0\}) \quad 1 \leq i \leq p$$

$$V_i'=\pi(\{x \in U \mid (x_{3i-2}, x_{3i-1})=(0,0)\})\ p+1 \leq i \leq n+1.$$

The points of S_i' satisfy Hausdorff axiom with every point of V_i', thus we get as in lemma 1′ that $\pi \circ f: S_i \to S_i'$ is a finitely-sheeted covering.

Consider a pair: $(I,J)=(\{i_1,\dots,i_r\},\{j_1,\dots,j_s\})$ $I \cup J \subsetneq \{1,\dots,n+1\}$ with

$1 \le i_1 < \cdots < i_r \le p$ and $p+1 \le j_1 < \cdots < j_s \le n+1$. There exists at least one component $S_{I\,J}$ of the set:

$$\left.\begin{array}{l} f_{3i_1} = \cdots = f_{3i_r} = 0 \\ (f_{3j-2}, f_{3j-1}) = (0,0) \quad j \in J \\ g_k \neq 0 \quad k \notin I \cup J \end{array}\right\}$$

contained in $\bar{C}$. We choose such an $S_{I\,J}$ for each pair (I, J).

Define $S'_{i_1,\ldots,i_r,j_1,\ldots,j_s}$ and $V'_{i_1,\ldots,i_r,j_1,\ldots,j_s}$ in the same fashion as S'_i, V'_i. We again see that $\pi \circ f : S_{I\,J} \to S'_{I\,J}$ is a finitely-sheeted covering. Let $q = n+1-p$, there is a homeomorphism:

$$h : Q_0 \cup \bigcup_{I,J} S'_{I\,J} \to (S^1)^p \times D_q^{q+n}$$

where D_q^{q+n} is the following $(q+n)$-cell:

$$D_q^{q+n} = \{(x_1, y_1, \ldots, x_q, y_q, z_1, \ldots, z_p) \in S^{q+n} \mid z_1, \ldots, z_p \ge 0\}.$$

EXAMPLE 1. For $n=1$ $q=1$, D_1^2 is the upper hemisphere of S^2.

EXAMPLE 2. For $n=2$ $q=0$, D_0^2 is the positive octant of S^2, it has boundary and corners.

EXCEPTION. If $p=0$, then $D_{n+1}^{2n+1} = S^{2n+1}$ is not a cell.

The set $h(S'_{I\,J})$ is:

$$(S^1)^p \times \{x \in D_q^{q+n} \mid (x_j, y_j) = (0,0) \Leftrightarrow j \in J \ z_i = 0 \Leftrightarrow i \in I\}$$

and $h(Q_0)$ is:

$$(S^1)^p \times \{x \in D_q^{q+n} \mid (x_j, y_j) \neq (0,0) \ \forall j \ z_i > 0 \ \forall i\} \approx T^{n+1} \times \mathring{D}^n.$$

It is easy to see now that if W is a tubular neighbourhood of $S'_{I\,J}$ in

$$Q_0 \cup \bigcup_{I',J'} S'_{I'\,J'} \text{ and } V = W \cap Q_0,$$

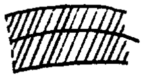
then the inclusion $V \hookrightarrow Q_0$ induces an isomorphism of the π_1-groups and $((\pi \circ f)|C)^{-1}(V)$ must be connected, since V is. Also note that if $p_1 = \# I$ then V is the interior of a corner subbundle of the tubular neighbourhood W:

$$W \approx S'_{I\,J} \times D^m \text{ for some } m; \quad V \approx S'_{I\,J} \times D^{m-p_1} \times (0, \varepsilon)^{p_1}$$

here $(0, \varepsilon)$ denotes an open interval.

Pictorially:

Note that $S'_{I\,J}$ is not compact, thus the tubular neighbourhood W has open ends:



We come now to the key point: in dim = 3, the pieces $S_{I\,J}$ were compact and letting N be a compact arbitrarily small tubular neighbourhood of $S_{I\,J}$ then $\pi\circ f(\partial N)$ was disjoint from $S'_{I\,J}$. Now the pieces $S_{I\,J}$ are not compact, but they are open submanifolds of compact submanifolds:

$$S_{I\,J}\subset H_{I\,J}=\{x\in M\,|\,f_{3i}(x)=0\ i\in I\ (f_{3j-2}(x),f_{3j-1}(x))=(0,0)\ j\in J\}.$$

The image $\pi\circ f(H_{I\,J})$ is apriori not necessarily Hausdorff (although it will turn out to be, as a consequence of the next lemma), nevertheless, we can still find a tubular neighbourhood N of $H_{I\,J}$ such that $\pi\circ f(\partial N)$ is disjoint from $\pi\circ f(H_{I\,J})$. To see this, consider the following subset of L:

$$H'_{I\,J}=\pi(\{x\in U\,|\,x_{3i}=0\ i\in I\ (x_{3j-2},x_{3j-1})=(0,0)\ j\in J\})$$

which is a disjoint union of compact submanifolds of several dimensions.

The "normal bundle" $\nu(H'_{I\,J})$ has fibres of different dimensions too.

By examining directly the position of $H'_{I\,J}$ in L, one sees there is a metric on TL and an $\varepsilon>0$ such that for the ε-disc bundle $\nu_\varepsilon(H'_{I\,J})$ the inverse image of $H'_{I\,J}$ under Exp: $\nu_\varepsilon(H'_{I\,J})\to L$ reduces to the zero section. Take now N in M so that $\pi\circ f(N)$ is covered by Exp $(\nu_\varepsilon(H'_{I\,J}))$.

For a subset A of C, let $Bd^C(A)$ denote the boundary of A in the subspace topology of C, then $Bd^C(A)\subset C\cap Bd(A)$. Suppose N open, then:

$$Bd^C(N\cap C)\subset C\cap Bd(N\cap C)=C\cap(\overline{N\cap C}-N\cap C)\subset C\cap(\bar N\cap\bar C-N\cap C)=$$

$$C\cap\bar N-C\cap N=C\cap(\bar N-N)=C\cap Bd(N).$$

This implies that $\pi\circ f(Bd^C(N\cap C))$ is disjoint from $\pi\circ f(S_{I\,J})=S'_{I\,J}$.

Choose a tubular neighbourhood W of $S'_{I\,J}$ disjoint from $\pi\circ f(\partial N)$. Then the inverse image $((\pi\circ f)|C)^{-1}(V)$ is disjoint from $Bd^C(N\cap C)$, which implies: $((\pi\circ f)|C)^{-1}(V)\subset N\cap C$, and N is arbitrarily small.

Let now (x_k) be a sequence in C and suppose that the sequence $y_k=\pi\circ f(x_k)$ is convergent in L. Since

$$C_0\cup\bigcup_{I,J}S'_{I\,J}$$

is compact (not closed!) in L, we can take the limit point in this subset:

$$y_k\to y\in C_0\cup\bigcup_{I,J}S'_{I\,J}.$$

If $y\in C_0$, then, by lemma 1′, (x_k) must accumulate to some $x\in C$.

Suppose $y\in S'_{I\,J}$. Then the above argument shows that x_k accumulates to some $x\in H_{I\,J}$; in fact, for this particular x:

$$(**)\qquad \begin{cases} f_{3i}(x)=0 & i\in I \\ (f_{3j-2}(x), f_{3j-1}(x))=(0,0) & j\in J \\ g_k(x)\neq 0 & k\notin I\cup J \end{cases}$$

otherwise we would have $y\in S'_{I'J'}$ with $I'\cup J'\supsetneq I\cup J$. But the S'_{IJ} are disjoint by definition, so y belongs to only one. Conditions (**) determine an open subset H^*_{IJ} of H_{IJ}, and it is a consequence of what we just saw that the inverse image of S'_{IJ} under $(\pi\circ f)|\bar{C}$ is the union of some connected components of H^*_{IJ}:

$$S_{IJ}=S^{(0)}_{IJ}, S^{(0)}_{IJ}, \dots, S^{(r)}_{IJ}.$$

On the other hand, if W is a tubular neighbourhood of S'_{IJ} in

$$Q_0\cup \bigcup_{I',J'} S'_{I'J'} \text{ and } V=W\cap Q_0,$$

then $V\cup S'_{IJ}$ looks like:

$$S'_{IJ}\cup(S'_{IJ}\times D^{m-p_1}\times(0,\varepsilon)^{p_1}),$$

which implies that $(\pi\circ f)^{-1}(V\cup S'_{IJ})$ looks locally the same, i.e. it is a locally trivial bundle over $(\pi\circ f)^{-1}(S'_{IJ})$ with fibre:

$$(D^{m-p_1}\times(0,\ )^{p_1})\cup\{0^m\}.$$

This bundle minus its zero section is totally contained in components C. All this, together with the fact that $C\cap(\pi\circ f)^{-1}(V)$ is connected, implies that $\bar{C}\cap(\pi\circ f)^{-1}(S'_{IJ})$ is connected, thus equal to S_{IJ}.

As a conclusion: if $x_k\in C$ and $\pi\circ f(x_k)\to x\in S'_{IJ}$, then x_k accumulates to some $x\in S_{IJ}$. In other words, the union:

$$C\cup \bigcup_{I,J} S_{IJ}$$

is compact. Thus it is the closure of C. We see that the map:

$$\pi\circ f:\bar{C}\to Q_0\cup \bigcup_{I,J} S'_{IJ}$$

is a covering as we did for $\dim=3$. QED

REMARK. The pieces $\bar{S}_k$ are connected, and if $n\geq 2$ then they intersect in $\bar{S}_{k,k'}$, thus ∂C is connected for $n\geq 2$.

We now state the fact that makes all the difference between the case $n=1$ and the case $n\geq 2$, for which the contact form is unique in the second and not in the first.

LEMMA 3. *Let M, f be as in theorem B, and $n\geq 2$. Then for every component C of $(\pi\circ f)^{-1}((S^0)^{n+1}\times Q)$ the set I_0 defined in lemma 2′ is the same.*

PROOF. Take a component C and define the number p as we did in the proof of lemma 2′.

If $p=0$, then $\pi\circ f:\bar{C}\to S^{2n+1}$ is a diffeomorphism and there is only one component C, we are done. Thus assume $p>0$.

For $p+1\leq j\leq n+1$, $\bar{S}_j$ satisfies the equation $(f_{3j-2}, f_{3j-1})=(0,0)$ and we see in the same way as for the circles in dimension three that $S_j \cup C$ is open. Thus the pieces of ∂C really are the $\bar{S}_i$ with $1\leq i\leq p$:

$$\partial C = \bigcup_{I\subset\{1,\dots,p\}} S_{I,\emptyset}.$$

Given the components C and C', we can find a path $\gamma(t)$ satisfying:

i) $\gamma(0)\in C$ $\gamma(1)\in C'$

ii) For every component C'' such that γ hits $\bar{C}''$, γ intersects $\partial C''$ transversally at one or two pieces of codimension 1.

Thus C, C' are joined by a finite chain of components:

$$C=C_0, C_1, \dots, C_r=C'$$

such that ∂C_k and ∂C_{k+1} share a $2n$-dimensional piece.

Let $S_1, S_2, \dots, S_{n+1}$ be the pieces of ∂C_k, here S_i is given by

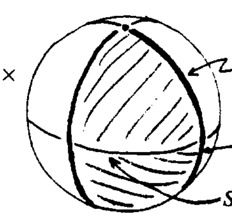
$$f_{3i}\cdot(f_{3i-2}, f_{3i-1})=(0,0).$$

Let $S_1, S_{(2)}, \dots, S_{(n+1)}$ be the same for ∂C_{k+1}.

Now, for $i\geq 2$ the pieces $\bar{S}_i$ and $\bar{S}_{(i)}$ both intersect $\bar{S}_1$ (would not be true for $n=1$) in the same points because of uniqueness of the $S_{I\,J}$'s in lemma 2'. Thus:

either a) $f_{3i}=0$ on S_i and $S_{(i)}$

or b) $(f_{3i-2}, f_{3i-1})=(0,0)$ on S_i and $S_{(i)}$



this proves the lemma.

PROOF OF THEOREM B. From lemma 3, we get a subset $I_0\subset\{1,\dots,n+1\}$ such that:

$$\left.\begin{array}{ll} (f_{3i-2}, f_{3i-1})\neq(0,0) & i\in I_0 \\ f_{3j}\neq 0 & j\notin I_0 \end{array}\right\}$$

in all of M. That means $f(M)\subset U_{I_0}$. Then there is a connected component M_1 of $\pi(U_{I_0})$ such that $\pi\circ f:M\to M_1$ is a covering.

By exchanging f_{3j-2}, f_{3j-1} for some j's in the complement of I_0, we can assume: $f_{3j}>0$ on M $\forall j\notin I_0$. Then the inclusion $M_1\hookrightarrow(U,\bar{\omega}_0)$, given by $M_1\subset V_{I_0}\subset U$, induces on M_1 the standard contact form of $T^p\times S^{n+q}$, where

$p=\#I_0$ $q=n+1-p$. Since $n+q\geq n\geq 2$ implies $\pi_1(S^{n+q})=0$, there is a diffeomorphism $\varphi: M\to T^p\times S^{n+q}$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[d, "\varphi"'] \arrow[dr, "\pi\circ f"] & \\ T^p\times S^{n+q} \arrow[r, "R\times id"'] & T^p\times S^{n+q} \end{tikzcd}$$

The map $R: T^p\to T^p$ is a standard covering, i.e. there is a matrix of integers:

$$\begin{pmatrix} a_{11} & & & 0 \\ a_{21} & a_{22} & & \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{pmatrix}$$

the diagonal ones positive, such that:

$$R(z_1,\ldots,z_p)=(z_1^{a_{11}}\cdot z_2^{a_{21}}\cdots z_p^{a_{p1}},\ z_2^{a_{22}}\cdots z_p^{a_{p2}},\ldots,z_p^{a_{pp}}).$$

Now, if ω_0 is the standard form on $T^p\times S^{n+q}$ we have like for the three-dimensional case: $f^*\omega_0=\varrho\varphi^*(R\times id)^*\omega_0$ for some $\varrho>0$. So we only need to show that $(R\times id)^*\omega_0$ is equivalent to ω_0.

It is possible to rearrange indices so that $I_0=\{1,2,\ldots,p\}$.

From:

$$\omega_0=\sum_{i\leq p} x_i d\theta_i+\sum_{i=1}^{q}(x_{p+2i-1}dx_{p+2i}-x_{p+2i}dx_{p+2i-1})$$

we get:

$$(R\times id)^*\omega_0=\sum_{i\leq p} x_i R^*d\theta_i+\sum_{i=1}^{q}(x_{p+2i-1}dx_{p+2i}-x_{p+2i}dx_{p+2i-1})=$$

$$=\sum_{k=1}^{p}\Big(\sum_{i=1}^{k} x_i a_{ki}\Big)d\theta_k+\sum_{i=1}^{q}(x_{p+2i-1}dx_{p+2i}-x_{p+2i}dx_{p+2i-1})=\bar g^*\bar\omega_0$$

where $\bar g: T^p\times S^{n+q}\to\mathbb{R}^{3n+3}$ is given by:

$$\bar g(\theta_1,\ldots,\theta_p,x_1,\ldots,x_{p+2q})=(\cos\theta_1,\ \sin\theta_1,y_1,\ldots,\ \cos\theta_p,\ \sin\theta_p,$$
$$y_p,y_{p+1},y_{p+2},1,\ldots,y_{p+2q-1},y_{p+2q},1)$$

$$\begin{pmatrix} y_1\\ \vdots \\ y_{p+2q}\end{pmatrix}=\left(\begin{array}{cccc|ccc} a_{11} & & & & & & \\ a_{21} & a_{22} & & & & & \\ \cdots & \cdots & \cdots & \cdots & & & \\ a_{p1} & a_{p2} & \cdots & a_{pp} & & & \\ \hline & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{array}\right)\begin{pmatrix} x_1\\ \vdots \\ x_{p+2q}\end{pmatrix}$$

The matrix being invertible, we have $\bar{g}(T^p \times S^{n+q}) \subset U_{I_0}$ and $\pi \circ \bar{g}: T^p \times \times S^{n+q} \to T^p \times S^{n+q}$ is a diffeomorphism. Thus $\bar{g}^* \bar{\omega}_0$ is equivalent to ω_0.

QED

BIBLIOGRAFY

1. Bennequin, D. – Entrelacements et équations de Pfaff. Astérisque, **107-108**, 87-161 (1983).
2. Cerf, J. – Sur les difféomorphismes de S^3. Lecture note series, Springer, **53**, (1968).
3. Godbillon, C. – Géométrie différentielle et mécanique analytique. Hermann, Paris, 1969.
4. Gonzalo, J. and F. Varela – Some surfaces that are distinguishable by the global expression of their volume element. Mathematische Annalen, Springer, **267**, 199-212 (1984).
5. Lutz, R. – Structures de contact sur les fibrés principaux en cercles de dimension 3. Ann. Inst. Fourier, Grenoble, XXVII, **3**, 1-15 (1977).
6. Lutz, R. – Sur la géométrie des structures de contact invariantes. Ann. Inst. Fourier, Grenoble, XXIX, **1**, 283-306 (1979).